Hydrodynamical activity in thin accretion disks

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Abstract

An asymptotic treatment of thin accretion disks, introduced by Kluźniak & Kita (2000) for a steady-state disk flow, is extended to a time-dependent problem. Transient growth of axisymmetric disturbances is analytically shown to occur on the global disk scale. The implications of this result on the theory of hydrodynamical thin accretion disks, as well as future prospects, are discussed.

Key words: accretion disks, hydrodynamical stability, transient growth

1. Introduction

Thin accretion disks (AD) form whenever a sufficiently cool gas, endowed with a significant amount of angular momentum, is gravitationally attracted towards a relatively compact object. This situation is quite common in astrophysics and therefore the observational and theoretical study of accretion disks has been quite intensive. The necessary condition for accretion to take place is that angular momentum be extracted from the fluid swirling around the central object. To achieve accretion rates that are consistent with observations, the physical mechanism for such angular momentum transport must be more efficient (by many orders of magnitude) than just the one resulting from torques caused by microscopic viscosity.

Already at the outset, when AD were theoretically proposed (Prendergast & Burbidge 1968, Pringle & Rees 1972), the enhanced effective viscosity was postulated to result from turbulence and was parametrized, using mixing length theory or a similar scheme, as detailed theoretical understanding of turbulence and the transition to it was lacking then (a situation that is still with us nowadays). The parametrization of the effective viscosity in disks in terms of a single parameter - α, introduced by Shakura & Sunyaev (1973), proved itself to be the most fruitful, giving rise to successful interpretations of many observational results (see Lin & Papaloizou 1996, Frank, King & Raine 2002, for reviews). In some cases (e.g. dwarf nova models), however, more complex (and cumbersome) prescriptions for the viscosity had to be employed. However, until the early 1990's the question what is the physical origin of turbulence (or, more precisely, the anomalous angular momentum transport) in AD has essentially remained unanswered. No definite linear instability has been identified for thin disks in which the swirling flow consist of Keplerian shear. The magneto-rotational instability (MRI), originally found (Velikhov 1959, Chandrasekhar 1960) for magnetic Taylor-Couette, (i.e., cylindrical) flows, has been shown by Balbus & Hawley (1991) to also operate in cylindrical AD, when the fluid is electrically conducting and for not too large initial magnetic fields.

It has thus become the operating paradigm in the astrophysical community that purely hydrody-

namical turbulence in Keplerian disks is altogether ruled out and since the problem of angular momentum transport in these objects relies on MRI driven magneto-hydrodynamical (MHD) turbulence, extensive numerical calculations are necessarily the main tool of research on this problem (see Balbus & Hawley 1998 and Balbus 2003, for reviews and references). Nonetheless, efforts to find a purely hydrodynamical transition to turbulent activity in thin AD have continued until the present time, more than a decade after Balbus , Hawley & Stone (1996) appeared to settle the matter. These have largely been motivated by the fact that a purely hydrodynamical AD flow has a 'microscopic' Reynolds number (Re) of the order of 10^{14} or so, a quite amazing setting for a laminar flow.

Among the ideas that have been put forward in this context some are based on the application to thin disk flows of the viewpoint, familiar to the fluid-dynamical community, that *transient dynamics induced by perturbations*, i.e. strong transient growth (TG) in linearly stable shear flows may play an important rôle in nonlinearly shaping the final dynamical state. TG is possible because the relevant linear operator governing the behavior of infinitesimal perturbations in these flows is non-normal and thus the usual modal approach essentially fails (see, e.g., Grossman 2000, Schmid & Hennigson 2001, and Criminale, Jackson & Joslin 2003)

TG of disturbances has been discussed in the astrophysical literature within two quite different (but related) contexts:

- Local disturbances experiencing large enough TG that can possibly trigger a subcritical nonlinear transition into turbulence in a linearly stable shear flow, i.e. via the so-called bypass transition (e.g. Chagelishvili *et al.* 2003, Tevzadze *et al.* 2003, Yecko 2004, Afshordi *et al.* 2005, Mukhopadhyay *et al.* 2005)
- Perturbations experiencing significant TG that can be excited either by some external agent, perhaps as secondary flows on a pre-existing 3D turbulence itself, however weak, may give rise to intense global dynamical activity (e.g. Ioannaou & Kakouris 2001, and see below).

Other scenarios for inducing hydrodynamical activity in AD (some of them directly or indirectly related to TG as well) have also been proposed. Among them are those invoking baroclinic instabilities (e.g., Klahr & Bodenheimer 2003), strato-rotational instabilities (e.g., Dubrulle *et al.* 2005, Umurhan 2006), formation and long sustenance of vortices and/or waves (e.g., Godon & Livio 1999, Bracco *et al.* 1999, Li *et al.* 2001, Umurhan & Regev 2004, , Barranco & Marcus 2005, Petersen, Stewart & Julien 2007, Bodo *et al.* 2007, Lithwick 2007). Even though no definite and undisputed hydrodynamical mechanism, that can give rise to sufficient angular momentum transport in AD, has been identified so far, significant efforts along these lines are continuously being made by several groups of researchers.

It appears that continued study of purely hydrodynamical processes in disks remains viable and worthwhile, in particular in view of the difficulties with MRI driven angular momentum transport in AD, which have recently been pointed out. Questions of insufficent numerical resolution in MHD disk simulations have been convincingly raised (Pesah, Chen & Psaltis 2007, Fromang & Papaloizou 2007). The decrease of transport with decreasing magnetic Prandtl number (Pm) for various setups and boundary conditions and, in particular, the vanishing of this transport for $Pm \ll 1$ (which is the case in many instances of astrophysical AD) has been demonstrated (Umurhan, Menou & Regev 2007, Lesur & Longaretti 2007, Umurhan, Regev & Menou 2007, Fromang *et al.* 2007). Finally, serious doubts as to the viability of the local shearing box approximation (Goldreich & Lynden-Bell 1965) to the numerical study of accretion disk MHD turbulence (Coppi & Keyes 2003, King, Pringle & Livio 2007, Shu *et al.* 2007, Regev & Umurhan 2007) have been raised.

The purpose of this contribution is to draw attention to the possibility that a thin AD, in which a sub-critical transition to hydrodynamical turbulence occurs (for a sufficiently high Re) large transient amplification of global disturbances may induce recurrent or even persistent secondary flow activity. Lesur & Longaretti (2005) have explicitly demonstrated, using high resolution 3D numerical simulations, that such sub-critical transition to turbulence does appear in a model flow having Keplerian shear characteristics, but at a very high Re (see also Rincon *et al.* 2006). However, it appears that the efficiency of turbulent transport in these flows is insufficient for astrophysical purposes (i.e. AD). A similar conclusion can perhaps be drawn from the recent experimental study of Ji *et al.* (2006). The very small value of the effective α (measuring the angular momentum transport) in these flows is, as we shall show, the key necessary ingredient for the excitation of vigorous global (transient) secondary

flows, atop the weak turbulent state. It is quite conceivable that the large (by orders of magnitude in the disturbance energy) TG may (nonlinearly) give rise to persistent dynamical activity, or at least be recurrently re-excited by external perturbations (see Ionnaou & Kakouris 2001, who advocated the latter possibility). It however remains to be shown, of course, that in the ultimate state angular momentum transport is appropriate for AD, i.e. the effective α acquires a high enough value.

The primary tool utilized here in order to facilitate an analytical treatment is the asymptotic expansion, where the dependent variables and governing equations are expanded in powers of a small quantity (here the measure of the disk's 'thickness', ϵ - see below). Exposing the resulting mathematical system (an initial value problem) to a set of initial conditions, in which the extreme geometry of the disk structure is taken into account (i.e. $\epsilon \ll 1$), we aimed at obtaining an analytically treatable problem. To achieve this goal we also assumed a polytropic relation between the pressure and density. The advantage of such an approach is obvious - the treatment can be essentially analytical and the responsible physical effects leading to any interesting dynamics may be transparently traced. The study reported on here is limited to axi-symmetric disturbances and the question of its ultimate development still remains open. We are now in the process of generalizing it to fully 3-D disturbances and it is clear that the present analytic work should be ultimately complemented with detailed and uncompromising 3-D numerical calculations with a proper treatment of energy generation and transfer.

2. Asymptotic formulation of the dynamics of a thin polytropic accretion disk

We wish to investigate the dynamical (i.e. time dependent) behavior of thin, axisymmetric AD, allowing for vertical structure. *Viscous* flow is invoked, with a standard viscosity prescription, that is we actually assume that an effective viscosity producing mechanism is already operative. The mean flow is thus assumed to consist of Keplerian thin disk flow, which has already undergone a sub-critical transition, like the one found by Lesur & Longaretti (2005), for example. As mentioned above, these authors found that the value of the effective α scales with the value of the sub-critical transition Reynolds number, Re_g, as $\alpha \sim 1/\mathrm{Re}_g$ and since they obtained

that the value of Re_g for Keplerian rotation flows is extremely high, they concluded that the α values that can be expected are much smaller than those necessary for angular momentum transport in astrophysical AD. He we shall examine the dynamical behavior for different values of α and, as we shall see, the TG we find will be anti-correlated with the value of α of the mean steady flow.

The key observation is that when on writes the equation of fluid dynamics (e.g., Tassoul 1978) and makes them non-dimensional by scaling the variables with their typical values, a non-dimensional parameter ϵ appears, multiplying (by its value in various powers) different terms. These (axisymmetric) equations in cylindrical coordinates (r, z, ϕ) read:

$$\partial_t \rho + \frac{\epsilon}{r}\partial_r(r\rho u) + \partial_z(\rho v) = 0, \tag{1}$$

$$\epsilon\partial_t u + \epsilon^2 u\partial_r u + \epsilon v\partial_z u - \Omega^2 r = -\frac{1}{r^2}\left(1 + \epsilon^2\frac{z^2}{r^2}\right)^{-\frac{3}{2}}$$
$$+\frac{\epsilon}{\rho}\partial_z\left(\eta\partial_z u\right) + \frac{\epsilon^2}{\rho}\left[\rho\partial_r\varpi + \partial_z\left(\eta\partial_r v\right) - \frac{2}{3}\partial_r\left(\eta\partial_z v\right)\right]$$
$$+\frac{\epsilon^3}{\rho}\left\{-\frac{2\eta u}{r^2} + \frac{2}{r}\partial_r\left(\eta r\partial_r u\right) - \frac{2}{3}\partial_r\left[\frac{\eta}{r}\partial_r(ru)\right]\right\}, \tag{2}$$

$$\partial_t v + \epsilon\partial_r v + v\partial_z v = -\frac{z}{r^3}\left(1 + \epsilon^2\frac{z^2}{r^2}\right)^{-\frac{3}{2}} - \partial_z\varpi$$
$$+\frac{4}{3\rho}\partial_z\left(\eta\partial_z v\right) + \frac{\epsilon}{\rho r}\left\{\partial_r\left(\eta r\partial_z u\right) - \frac{2}{3}\partial_z\left[\eta\partial_r(ru)\right]\right\}$$
$$+ \epsilon^2\frac{1}{\rho r}\partial_r\left(\eta r\partial_r v\right), \tag{3}$$

$$\partial_t\Omega + \epsilon\frac{u}{r^2}\partial_r(r^2\Omega) + v\partial_z\Omega =$$
$$+\frac{1}{\rho}\partial_z\left(\eta\partial_z\Omega\right) + \frac{\epsilon^2}{\rho r^3}\partial_r\left(\eta r^3\partial_r\Omega\right). \tag{4}$$

In these equations u, v and Ω are the horizontal, vertical and angular velocity respectively, $\varpi \equiv nc_s^2 = (n + 1)\rho^{1/n}$ (c_s is the sound speed and the non-dimensional polytropic relation $P = \rho^{1+1/n}$, with the general polytropic index n, has been used). Also, η is the dynamic viscosity, which can be expressed using the standard Shakura-Sunyaev viscosity prescription

$$\eta = \frac{2}{3}\frac{\alpha P}{\Omega_{\mathrm{K}}} = \frac{2}{3}\alpha\,\rho^{1+1/n}\,r^{-3/2}, \tag{5}$$

with the non-dimensional Keplerian angular velocity substituted as $\Omega_K = r^{-3/2}$.

The non-dimensional parameter $\epsilon \equiv \tilde{c}_s/(\tilde{\Omega}\tilde{r}) = \tilde{h}/\tilde{r}$ (the quantities denoted by tilde are the corresponding dimensional quantities, evaluated at a typical position in the AD, with $\tilde{h}$ being the disk height) is very small for thin disks (that is, those assumed to be able to cool efficiently, so that the rotational velocity is highly supersonic).

An asymptotic approach of the kind used here was introduced for the first time to the study of thin viscous AD by Regev (1983), in the context of AD boundary layers and later was developed and used in a remarkable analytical work of Kluźniak & Kita (2000), (hereafter KK) who solved for the *steady* structure of a polytropic viscous axisymmetric disk. This study revealed the presence of a steady meridional flow pattern with backflow for values of the α less than some critical value. This result and feature was confirmed by Regev & Gitelman (2002), who abandoned the polytropic assumption and included an energy equation (employing the diffusion approximation in the treatment of the vertical radiative transport), showing that the polytropic assumption makes only very little substantive difference from the *steady* meridional flow solution of KK.

The next step of the procedure consists of expanding all dependent variables asymptotic series in ϵ, for example

$$\Omega = \epsilon\Omega_0 + \epsilon^2\Omega_2 + \epsilon^4\Omega_4... \qquad (6)$$

Similar expansions are used for ρ and v, while the one for u consists of only odd powers of ϵ. The feasibility of this choice is justified in Umurhan *et al.* (2006) (hereafter UNRS). These expansions are then substituted into the full equations (1-4) and the expression in various orders in ϵ are collected. The resulting equations are similar to the ones found by KK, but there are some important differences:

(i) Time-dependence is included so as to be able to study the dynamics and consequently time-derivatives of the dependent variables appear (the time unit taken as the typical rotation time, $\tilde{t} = 1/\tilde{\Omega}$). The problem is thus formulated as an *initial value* problem.

(ii) Time-dependence is excluded in the lowest order term in the expansions and it is introduced only in the next significant order, in the form of an *additive* term, denoted by a prime, and can be viewed as a perturbation on the steady-state, but not necessarily an infinitesimal one)

Thus the typical even and odd order expansions are of the form

$$\rho(r, z, t) = \rho_0(r, z) + \epsilon^2 \left[\rho_2(r, z) + \rho_2'(r, z, t)\right]$$
$$+ \epsilon^4 \left[\rho_4(r, z) + \rho_4'(r, z, t)\right] + \cdots \qquad (7)$$
$$u(r, z, t) = \epsilon \left[u_1(r, z) + u_1'(r, z, t)\right] +$$
$$+ \epsilon^3 \left[u_3(r, z) + u_3'(r, z, t)\right] + \cdots \qquad (8)$$

(iii) As in KK the solution of $\mathcal{O}(1)$ equations yield a steady disk of the Shakura & Sunyaev kind, but for a polytrope (see also UNRS for the case of an arbitrary polytropic index n). The $\mathcal{O}(\epsilon)^2$ equations can be conveniently split into a steady and time-dependent parts, owing to the additive construction of the time-dependent disturbances. The steady part yields the KK steady solution, with possible backflows (which occur for $\alpha < \alpha_c \sim 0.7$).

We conclude this section by formulating the initial value problem resulting from the $\mathcal{O}(\epsilon)^2$ time-dependent equations.

$$\partial_t \rho_2' = -\frac{1}{r}\partial_r(r\rho_0 u_1') - \partial_z(\rho_0 v_2'), \qquad (9)$$

$$\partial_t u_1' = 2\Omega_0\Omega_2' r + \frac{1}{\rho_0}\partial_z\left(\eta\partial_z u_1'\right), \qquad (10)$$

$$\partial_t v_2' = -\partial_z \varpi_2' + \frac{4}{3\rho_0}\partial_z\left(\eta\partial_z v_2'\right) + \frac{1}{\rho_0 r}\partial_z\left(\eta r \partial_z u_1'\right)$$
$$- \frac{2}{3\rho_0}\partial_z\left\{\eta\left[\frac{1}{r}\partial_r(r u_1')\right]\right\}, \qquad (11)$$

$$\partial_t \Omega_2' = -\frac{u_1'}{r^2}\frac{d(r^2\Omega_0)}{dr} + \frac{1}{\rho_0}\partial_z\left(\eta\partial_z\Omega_2'\right), \qquad (12)$$

where the zero-indexed quantities are known from the steady $\mathcal{O}(1)$ (Shakura-Sunyaev disk) solution (e.g., $\Omega_0 = r^{-3/2}$) and we base the expression for η on the $\mathcal{O}(1)$ solution only, which using (5) is $\eta = (2/3)\alpha \; \rho_0^{1+1/n} \; r^{-3/2}$. In addition, from the polytropic law it follows that $\varpi_2' = \varpi_0\rho_2'/\rho_0$.

The initial value problem defined by (9-12) must be complemented by boundary conditions. We have opted for retaining the integral steady mass inflow condition as in KK. In addition, guided by the physical consideration that there should be no work done on the disk surface, we have required that the external pressure and viscous stresses vanish at this surface. For details see UNRS.

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3. Representative solutions

Consider now solutions to the initial value problem, posed in the previous section and for the sake of simplicity we set, from here and on, the polytropic index to the value $n = 3/2$ (as in KK). All solutions obviously depend on the initial conditions and our purpose here is to discuss some relevant representative solutions, that can exhibit TG. It is important to note that the above four equations support two types of solutions, namely a pair comprising of purely vertical disturbances and a second pair containing disturbances in all the velocity components. This follows from the fact that equations (10) and (12) dynamically decouple from the other two, i.e., the quantities u_1' and Ω_2' are not dynamically influenced by v_2' and ρ_2', while the converse is not true. Thus, in principle, if perturbations are chosen such that $u_1' = \Omega_2' = 0$ initially, they will remain so at all times and the motion will be of purely vertical acoustics (VA), as embodied in v_2' and ρ_2'. On the other hand, if one chooses non-vanishing u_1' & Ω_2' as initial conditions, the resulting dynamics comprises of all variables. We shall call these solutions "driven" general acoustics (GA). This mathematical structure is reminiscent of the behavior of the steady-state equations of KK and we shall now elaborate on these two solution types.

3.1. *Meridional velocity disturbances*

It is quite straightforward to derive from the differential set (9-12) two second order equations for just the function $u_1'(r, z, t)$ and $v_2'(r, z, t)$, that is, the lowest order time-dependent meridional components of the velocity perturbations. They are

$$\mathcal{P} u_1' = 0, \tag{13}$$

$$\mathcal{L} v_2' = [\partial_t \mathcal{F} + \mathcal{G}] \, u_1', \tag{14}$$

where the differential operators are defined (as operating on an arbitrary function $\varphi(r, z, t)$) in the following way

$$\mathcal{P} \equiv \left[\partial_t - \frac{1}{\rho_0} \partial_z(\eta \partial_z) \right] \left[\partial_t \varphi - \frac{1}{\rho_0} \partial_z(\eta \partial_z \varphi) \right] + \\ + \Omega_0^2 \varphi,$$

$$\mathcal{L} \equiv \partial_t^2 \varphi - \frac{2}{3} \varpi_0 \partial_z^2 \varphi - \frac{5}{3} \partial_z (\varpi_0 \partial_z \varphi) - (\partial_z^2 \varpi_0) \, \varphi - \\ - \frac{4}{3\rho_0} \partial_z [\eta \partial_z (\partial_t \varphi)],$$

$$\mathcal{F} \equiv -\frac{2}{3\rho_0} \partial_z [\eta \partial_r (r\varphi)] + \frac{1}{r\rho_0} \partial_r (r\eta \partial_z \varphi),$$

$$\mathcal{G} \equiv \partial_z \left[\frac{2\varpi_0}{3r\rho_0} \partial_r (r\rho_0 \varphi) \right]. \tag{15}$$

The boundary conditions completing this system can be specified using the 'free boundary' prescription at the disk surface, as mentioned at the end of the previous section (see UNRS for an explicit formulation). After solutions to equations (13-14) are found, it is possible to return to the original set of equations in order to find the two additional unknown functions $\rho_2'(r, z, t)$ and $\Omega_2'(r, z, t)$.

We were able to find analytical solutions to the system (13-14). In describing them we shall first discuss, in subsection 3.2, the case $u_1' = 0$ and $\Omega_2'=0$, which result from the special initial conditions in which these perturbations are not excited. This will thus allow us to be limited to solving only the homogeneous part of (14). The solutions of this equation constitute the above mentioned first pair of modes - VA. The solutions to the full set (13-14) in the general (i.e. inhomogeneous case, with $u_1' \neq 0$) will be discussed in subsection 3.3. This pair of modes was called before GA and we remark once again that their dynamics can be viewed as being driven (by the inhomogeneous part). Finally, in subsection 3.4 we shall describe in some detail the TG behavior found in our solutions. A full exposition of the calculations leading to the solutions quoted and discussed in subsections 3.2-3.4 is presented in UNRS and its Appendices. In the following three subsections only the salient features will be given.

3.2. *Vertical acoustics (VA)*

The general solution to the inhomogeneous equation (14) can be written, in the form

$$v_2' = v_h' + v_p' \tag{16}$$

where the indexes h and p stand for a solution of the homogenous equation and a particular solution of the inhomogeneous equation, respectively. As we have already remarked, the thinness of the disk allows for the density and vertical velocity fluctuations not to induce either radial or azimuthal motions at these orders. This implies that homogeneous solutions of (14), (i.e. with the equation's RHS set to zero, because $u_1' = \Omega_2' = 0$ say, see above) are perfectly acceptable.

These VA solutions have the form (see UNRS)

$$v_h' = \hat{v}_h'(z, r) \exp(\sigma_{\mathrm{v}} T) + \text{c.c.} \quad \text{with} \quad T \equiv \frac{t}{r^{3/2}}, (17)$$

where the spatial eigenfunctions $\hat{v}_h'$ are composed of the associated Legendre functions. This form of v_h' already indicates that these solutions are *inseparable* in r and t. The eigenvalues, which appear in complex conjugate pairs, are functions of α and in general they also depend, and quite sensitively, on the index n (see UNRS), but as mentioned before we concentrate here on the case $n = 3/2$ only. The eigenfrequency of the fundamental mode, for our chosen polytropic index is given by

$$\sigma_{\mathrm{v}} = -\frac{4}{9}\alpha \pm i \left| \left(\frac{16}{81}\alpha^2 - \frac{8}{3} \right) \right|^{1/2}. \tag{18}$$

All such eigenfrequencies have a negative real part and thus, as can be expected, the fundamental as well as all overtones, show temporal decay.

3.3. *General acoustics (GA)*

We turn now to the general solution to (13-14) by first focusing on (13). This equation admits an infinite set of eigenmode solutions in a similar way to the homogeneous solutions discussed before. The general derivation and structure of these eigenmodes is detailed in UNRS. For the sake of clarity, the discussion here focuses only upon the dynamics associated with the fundamental mode of the operator $\mathcal{P}$. This solution is given by (see UNRS)

$$u_1' = \hat{u}'(z, r) \exp(\sigma_{\mathrm{u}} T + \vartheta) + \text{c.c.}, \tag{19}$$

with $T \equiv t/r^{3/2}$, as before, and in which the spatial eigenfunction $\hat{u}'(z, r)$ has the particularly simple structure

$$\hat{u}'(r, z) = A(r) \left[\frac{z^2}{h^2} - \frac{1}{6} \right], \tag{20}$$

where h, the disk thickness is a known function of r, see (22). $A(r)$ is an amplitude whose radial functional form is technically arbitrary and would be set by the initial condition. An arbitrary phase factor of ϑ has also been introduced. The eigenvalues, denoted in this case by σ_u, again come in complex conjugate pairs and the fundamental frequency is given by

$$\sigma_{\mathrm{u}} = -\tfrac{8}{5}\alpha \pm i. \tag{21}$$

Without loss of generality we may only consider the '+' solution since the phase factor ϑ can account

for the '−' solution. The temporal behavior of these modes is again one of decaying oscillations, but with frequencies given by the local rotation rate of the disk. Note that in the prescription for $h(r)$ as given in UNRS for values of r sufficiently greater than the zero-torque radius, r_*, (see KK) the disk height with respect to r is well approximated by,

$$h(r) \sim h_1 r, \quad \text{and thus} \quad \frac{dh}{dr} \sim h_1, \tag{22}$$

where h_1 is determined by the mass flux rate and the value of α. Since h_1 simply adds a multiplicative factor to all the dynamical quantities, it plays no role in the quality of the ensuing evolution and, as such, we set $h_1 = 1$ without any loss of general flavor.

With this solution for u_1' we may find a particular solution to the vertical velocity, v_p' by solving (14), together with (9) directly. The details of this, rather lengthy procedure are presented in UNRS, but we can highlight the major steps here. We posit the following Ansatz for v_p' and ρ_p',

$$v_p' = \hat{v}_p(\zeta, r)e^{\sigma_{\mathrm{u}}T} + \hat{V}_p(\zeta, r)\left(Te^{\sigma_{\mathrm{u}}T}\right) + \text{c.c.}, \tag{23}$$

$$\rho_p' = \hat{\rho}_p(\zeta, r)e^{\sigma_{\mathrm{u}}T} + \hat{R}_p(\zeta, r)\left(Te^{\sigma_{\mathrm{u}}T}\right) + \text{c.c.}, \tag{24}$$

where $\zeta \equiv z/h(r)$.

Recall that we solve only for the fundamental mode of the system (see UNRS) and that there is obviously a dependence on α. It is important to notice the explicit appearance of a multiplicative T term in the above expressions. This temporal functional dependence is necessary in order to balance terms arising on the RHS of (14)- the 'driving' term and will, of course, cause TG (see below in the next subsection). In general, the form of these lowest order structure eigenfunctions is

$$\begin{pmatrix} \hat{v}_p \\ \hat{V}_p \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \zeta + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \zeta^3, \tag{25}$$

namely, it is a polynomial in ζ of the third degree with only odd powers in ζ. The four coefficients are functions of r and α, i.e. $a_i = a_i(r, \alpha)$ and $b_i = b_i(r, \alpha)$.

The solution to the density perturbation, $\hat{\rho}_p'$ can be quite straightforwardly obtained using (20), (23) and (24), taking the functional form of ρ_0 as detailed in the $\mathcal{O}(1)$ steady-state solution (see KK, UNRS) and finally integrating equation (9) with respect to time. This gives

$$\begin{pmatrix} \hat{\rho}_p \\ \hat{R}_p \end{pmatrix} = \left(1 - \zeta^2\right)^{\frac{1}{2}} \times$$

$$\left[\begin{pmatrix} c_0 \\ d_0 \end{pmatrix} + \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} \zeta^2 + \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} \zeta^4 \right], \quad (26)$$

where, again, the coefficients are functions of r and α, that is, $c_i = c_i(r, \alpha)$ and $d_i = d_i(r, \alpha)$.

The polynomials appearing in the square brackets of the expressions for $\hat{\rho}_p$ and $\hat{R}_p$ have only even powers of ζ. The detailed forms of the coefficients depend on the form of the generalized initial perturbation radial structure $A(r)$ (as well as on n, in the general polytrope case). We avoid presenting them here because they are very long and cumbersome; and the details of the coefficients have no effect on the resulting transient dynamics, since they describe only a particular type of disturbance. For the sake of simplicity most analytic results which will be presented below assume $A = e^{i\pi/4}$ (that is, a particularly simple form of the initial conditions with no dependence of the disturbance). We shall however show also one instance of a more complicated initial condition consisting of a localized Gaussian profile in r.

3.4. Transient dynamics and growth

As is evident by inspection of (23), v'_p and ρ'_p exhibit a pronounced TG during a finite time interval (due to the factor T multiplying the decaying exponential), before eventually decaying. To demonstrate this more clearly we may consider some integrated energy quantities. The first of these is an energy density, per unit *surface* of the disk

$$\mathcal{E}_a(r, t; \alpha) \equiv h(r) \int_{-1}^{1} \left(\tfrac{1}{2} \rho_0 v'^2_p + \tfrac{1}{2} \frac{c_{s0}^2 \rho'^2_p}{\rho_0} \right) d\zeta, \quad (27)$$

$\mathcal{E}_a(r, t; \alpha)$ is to be interpreted as the acoustic energy (per unit area of the disk) consisting of the kinetic energy in the vertical velocity disturbances and the compression energy due to the density disturbances. The contribution of the (purely oscillatory) velocity components, resulting from the homogeneous part of the disturbance equations, are left out of (27).

$\mathcal{E}_a$ depends on the α parameter as well as the structure of the radial velocity perturbation $A(r)$ (cf. 20). The choice of $A(r)$ as constant with respect to r makes the integral defined in (27) analytically
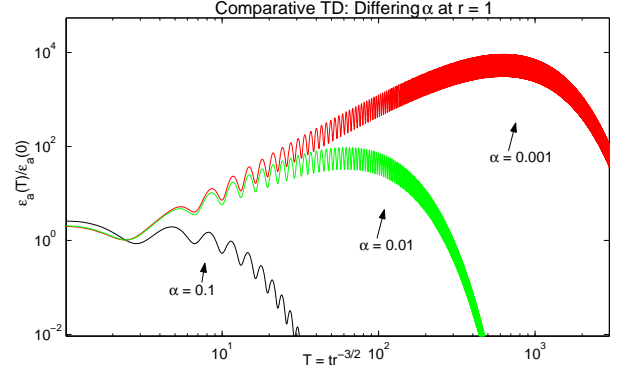

Fig. 1. The TG in the quantity $\mathcal{E}_a$ for the fundamental mode at $r = 1$, $n = 3/2$ and where $A(r) = e^{i\pi/4}$, $dA/dr = 0$. The four curves correspond to $\alpha = 0.1, 0.01, 0.001$ and are presented on a log-log plot. All curves are $\mathcal{E}_a$, scaled to its value at $T = 0$. The rise time, as well as the maximal value achieved, are proportional to the inverse of α. $\mathcal{E}_a$ also exhibits oscillations with period π that sit atop the general transient trend.

tractable. Nonetheless the expressions, which have been verified with the aid of Mathematica 5.0, are very long and will not be displayed here; only their essential features will be addressed. With $A = e^{i\pi/4}$, $\mathcal{E}_a$ has the functional form

$$\mathcal{E}_a(r, t; \alpha) = \frac{e^{-2\alpha T}}{r^{3/2}} \Upsilon(T, \cos 2T, \sin 2T; \alpha), \quad (28)$$

where Υ is a well-defined analytical, albeit complicated, function of its arguments. Thus, $\mathcal{E}_a$ depends on time only through the variable $T = t/r^{3/2}$, which is the time measured in units of the local disk rotation period, at r, divided by 2π.

In Figure 1, the evolution of $\mathcal{E}_a$ as a function of the variable T is shown for different values of α. T is actually a similarity variable and the value of r explicitly appears only in the pre-factor multiplying the function Υ. For the case displayed in the figure the radius is fixed (set to $r = 1$). Different values of r will merely change the overall height of the response ($\propto r^{-(3/2)}$) while the shape of the function is self-similar and is always decaying at long times T. Inspection of the figure, uncovers a very important property of the TG. As α decreases, the magnitude of the growth becomes more prominent [1] (for example, it is up to ~ 3 orders of magnitude for $\alpha = 0.001$) and the maximum occurs at correspondingly

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[1] For an inviscid disk, i.e. with formally $\alpha = 0$, Umurhan & Shaviv (2006) obtained algebraic growth with no decay, which is quite telling, but should be considered carefully (see Sec. 4 on the validity of asymptotic expansions).
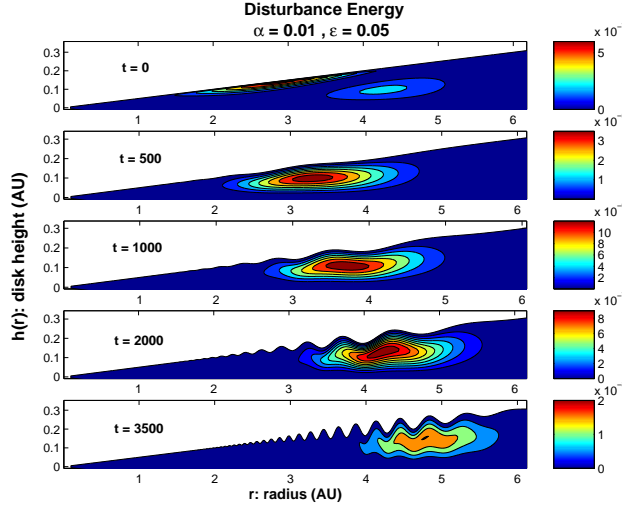
Fig. 2. TG in the vertical velocity disturbance kinetic energy density in a disk with $\alpha = 0.01, \epsilon = 0.05$. $A(r)$ is as in (30) with $r_0 = 3$ and $\Delta = 2.5$. Time is given in terms of disk rotation times at the radius $r = 1$, so that $T = t$. The magnitude of the disturbance energy density is indicated by the shadings (whose meaning is indicated on the right of each panel) on a cut through the disk's meridional plane. Significant growth in the disturbance energy is observed (almost three orders of magnitude). Crenellation patterns that appear in the fluctuating boundary of the disk are ultimately wiped out by viscous decay.

later times. The time corresponding to the maximum amplitude $T_{\max}$ is roughly $\sim 5/(8\alpha)$ in this[2] case. The rise in energy is modulated by oscillations which arise from the fact that there are correlations between pairs of variables which contribute to the overall growth in $\mathcal{E}_a(T)$. These variables oscillate in a frequency defined by the imaginary part of the eigenvalue (21) and because $\mathcal{E}_a(T)$ is composed of products of pairs of dynamical variables terms depending on $2T$ appear - explaining why the observed period is half the orbital period at a given radius.

$\mathcal{E}_a$ is a function of the radial position r, but one may form the integral quantity $E_a(t; \alpha)$, i.e., the *total disturbance acoustic energy* contained in a global portion of the disk (a ring), by integrating over the radial range in which these disturbances are assumed to exist, that is, between the inner and outer bounds $r_{\min}$ and $r_{\max}$ of the ring.

$$E_a(t; \alpha) \equiv \int_{r_{\min}}^{r_{\max}} \mathcal{E}_a(r, t; \alpha) \, 2\pi r \, dr. \qquad (29)$$

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[2] For the k-th overtone, $T_{\max}$ will be smaller by a factor of $1/k^2$, see UNRS)

We have found (see UNRS) that $A(r)$ affects the spatial details but not the global time behavior and that the occurrence of the peak energy value is delayed for larger values of $r_{\max}$, at given values of α (and $r_{\min}$), although the value of the energy at the peak time remains roughly the same for a given α. To get an idea about the expected qualitative *spatio-temporal* behavior of such transiently growing solutions we have computed the results for a case in which $A(r)$ is not constant (and, thus, the possibility to evaluate the relevant integrals analytically can not be expected, in general). We consider here an initial condition in which the radial velocity amplitude has a Gaussian form around some fixed radius, r_0 say, and having a width Δ, such that it is well contained within the region $r_{\min} \leq r \leq r_{\max}$

$$A(r) = e^{i\pi/4} \exp\left[-\frac{(r - r_0)^2}{\Delta_f}\right]. \qquad (30)$$

The fluid is assumed to be otherwise initially undisturbed. As pointed out above, with this form for A we lose the ability to find an analytic solution and must resort to numerical evaluation. In Figure 2 we show, in a contour plot in the disk meridional plane, the kinetic energy (per unit volume) contained in the vertical velocity disturbance v_p. The TG and decay of the disturbance is shown in the time sequence of figures displaying the spatial structure of the disturbance. We also depict how the disk surface moves in response to the imposed perturbation by solving the equation of motion for the boundary at this order. Time is given in units of rotation times of the disk, as measured at the radius $r = 1$. Because the response of the disk surface is an integral with respect to time, it is not a surprise to see a large amplitude in the surface position long after the kinetic energy has started to die away.

4. Summary and Discussion

We conclude with some remarks on possible improvements to the asymptotic analysis of the sort done here and prospects for the future, e.g., extensions to non-axisymmetric perturbations. Asymptotic expansions, when viable, are often very robust and provide a good approximation to the solution when truncation to only few leading terms is done. Obviously, when a term in the series becomes *very* large it may 'break its order', that is, become larger than a previous term and as such make the expansion invalid in this region. In our expansions succes-

sive terms ratios are of $\mathcal{O}\left(\epsilon^2\right)$, and thus the procedure's validity should not severely be limited even up to a growth factor of ~ 1000 or so (in the velocity or density perturbations). This matter is further discussed in UNRS, here we just remark that the validity with respect to time for weakly viscous solutions are somewhat influenced by the thinness of the disk: smaller values of ϵ mean that the solutions are valid for longer times after the initial disturbance. More importantly, however, for a given value of ϵ one must not be too zealous or overreaching by attempting to infer the quantitative behavior of the disk for arbitrarily small values of α - which, as one will recall, is formally assumed here to be an $\mathcal{O}\left(1\right)$ quantity.

Despite these caveats, the procedure when carried to higher order introduces corrections which are technically non-linear. Careful consideration must be undertaken in order to handle the response at these higher orders. This may entail treating the disturbance amplitudes for the lower order solutions (like $A(r)$ in u_1') as *weakly non-linear* governed by a second 'slow' time (e.g. the amplitude is instead written as $A = A(r, \tau)$ where $\tau = \epsilon^2 t$) in a manner analogous to the treatment of non-linear thick polytropes (e.g, Balmforth & Spiegel, 1996).

The approach used here may be generalized in a number of directions. Allowing for non-axisymmetric perturbations, including the disk inner and outer boundary in some kind of boundary layer analysis and relaxing the polytropic assumption seem to be the most obvious generalizations. We have found the presence of prominent TG in the simplest cases. It is difficult to imagine that it will be suppressed in the more general conditions although the effect of radiative energy losses on TG must be carefully examined.

The question concerning the ultimate fate of the transiently grown perturbations and their ability to induce a state of sustained complex dynamical activity in the disk remains open. In this context it is worthwhile to notice that since the TG decay times are of the order of hundreds of rotation periods, it is conceivable that AD, which are usually not isolated systems, may experience recurrent external perturbations on such time scales and in this way the dynamical activity may be sustained. Extensive numerical calculations of AD are however needed to decide if TG may lead, through non-linear processes, to sustained turbulence, or at least a dynamical state in which adequate angular momentum transport can be sustained. Such high-resolution global 3-D calculations are, however, still above the ability of the present computer power and it may be thus advantageous to also exploit sophisticated non-linear asymptotic methods to complement and guide them.

5. Afterword

Accretion disks were introduced to explain astronomical X-ray sources which were discovered close to four decades ago, when Jean-Pierre Lasota was still a graduate student. At the time of the conference held at the Trzebieszowice Castle, on the occasion of his 65-th birthday, it seemed quite obvious that although we have gone a long way towards understanding the physics of these fascinating and astronomically ubiquitous objects, much still remains to be elucidated. In order to enable accretion rates that are compatible with observations of such diverse objects as forming stars, close binary systems or active galactic nuclei, an efficient physical mechanism for angular momentum transport must be present in these AD flows. It has been recognized at the outset that fluid turbulence offers a natural mechanism of this sort, but hydrodynamical (and also MHD) turbulence remains, unfortunately, until this day to be one of the major bugbears of theoretical physics.

Shear flows are known to be particularly difficult in trying to understand the physics of transition to turbulence in them. Even the simplest flows of this kind, like the plane Couette and Poiseulle ones and the pipe flow, are not well understood in this respect (see, e.g., Drazin 2002). How, then, can we reasonably expect to adequately understand this problem in AD, with their literally astronomical Re, strong shear and rotation, compressibility, magnetic fields permeating an ionized medium and other physical complications? The lesson learned from stellar turbulent convection is that the crudest (e.g, mixing-length) phenomenological approaches may be very effective in constructing observationally viable models of stars, but detailed understanding is extremely difficult.

As far as AD are concerned, making significant progress beyond the famous α model requires, at least in my view, a combined synergetic effort of different groups of researchers, approaching the problem from many different angles and employing essentially all relevant methods that theoretical physics can offer. Phenomenology, analytical and semi-analytical investigation of simplified systems, laboratory experiments and numerical simulations

of various kinds can all be useful, and especially if they fertilize each other. Paradigms have always played an important rôle in scientific progress, often consecutively replacing one another. Dogmas, however, have always been stumbling blocks, especially in the education and mentoring of young researches who are naturally able to offer fresh and independent ideas.

If we are able to avoid such dogmas and truly cooperate in the way hinted on above, there are good chances that the physics of accretion disks will be much better understood when we celebrate Jean-Pierre's proverbial 120-th birthday, and hopefully even much sooner.

Acknowledgements

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